

DETERMINATION OF MINIMUM LENGTH OF SOME LINEAR CODES

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ABSTRACT. Hamada ([8]) and Maruta ([17]) proved the minimum length $n_3(6, d) = g_3(6, d) + 1$ for some ternary codes. In this paper we consider such minimum length problem for $q \geq 4$, and we prove that $n_q(6, d) = g_q(6, d) + 1$ for $d = q^5 - q^3 - q^2 - 2q + e$, $1 \leq e \leq q$. Combining this result with Theorem A in [4], we have $n_q(6, d) = g_q(6, d) + 1$ for $q^5 - q^3 - q^2 - 2q + 1 \leq d \leq q^5 - q^3 - q^2$ with $q \geq 4$. Note that $n_q(6, d) = g_q(6, d)$ for $q^5 - q^3 - q^2 + 1 \leq d \leq q^5$ by Theorem 1.2.

1. Introduction

Let \mathbb{F}_q denote the Galois field of q elements and \mathbb{F}_q^n denote the n -dimensional vector space over \mathbb{F}_q , where q is a prime power. For a vector $x = (x_1, \dots, x_n) \in \mathbb{F}_q^n$, the weight of x denoted by $w(x)$ is the number of nonzero coordinates of x , that is, $w(x) = |\{i \mid x_i \neq 0\}|$.

An $[n, k, d]_q$ linear code C is a k -dimensional subspace of \mathbb{F}_q^n over \mathbb{F}_q with minimum distance d . One of the central problems in coding theory is to determine optimal linear codes. This is to optimize one of the parameters n , k and d for given the other two as follows; (1) Find the smallest length n , denoted by $n_q(k, d)$, for which there exists an $[n, k, d]_q$ code for given k and d . (2) Find the largest minimum distance d , denoted by $d_q(n, k)$, for which there exists an $[n, k, d]_q$ code for given k and n . (3) Find the largest dimension k , denoted by $k_q(n, d)$, for which there exists an $[n, k, d]_q$ code for given n and d .

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A code of length $n_q(k, d)$ [resp. minimum distance $d_q(n, k)$, dimension $k_q(n, d)$] is said to be length-optimal [resp. distance-optimal, dimension-optimal]. Note that a length-optimal code is both distance-optimal and dimension-optimal. So we concentrate on the length-optimal codes. The following is an important lower bound on $n_q(k, d)$ which is called the Griesmer bound.

THEOREM 1.1 ([9]). (Griesmer bound) For an $[n, k, d]_q$ linear code, we have $n \geq g_q(k, d)$, where $g_q(k, d) = d + \left\lceil \frac{d}{q} \right\rceil + \left\lceil \frac{d}{q^2} \right\rceil + \dots + \left\lceil \frac{d}{q^{k-1}} \right\rceil$.

By Theorem 1.1, we note $n_q(k, d) \geq g_q(k, d)$ for all k and d . It is natural to ask whether there exists a $[g_q(k, d), k, d]_q$ code for given d and k . The following theorem gives a large class of linear codes meeting the Griesmer bound which we call Griesmer codes.

THEOREM 1.2 ([9]). Let $s = \left\lceil \frac{d}{q^{k-1}} \right\rceil$ and $d = sq^{k-1} - \sum_{i=1}^p q^{u_i-1}$ with $k > u_1 \geq u_2 \geq \dots \geq u_p$ and $u_i > u_{i+q-1}$ for $1 \leq i \leq p - q + 1$. If

$$\sum_{i=1}^{\min\{s+1, p\}} u_i \leq sk,$$

then $n_q(k, d) = g_q(k, d)$.

Theorem 1.2 provides a starting point for finding $n_q(k, d)$. For $k = 1$ and 2 , we have $n_q(k, d) = g_q(k, d)$ for all d . Thus we are interested in $k \geq 3$.

From Theorem 1.2, we have the following:

COROLLARY 1.3 ([5]). We have $n_q(k, d) = g_q(k, d)$ for d satisfying one of the following:

- (a) $q^{k-1} - q^{k-1-t} - q^t + 1 \leq d \leq q^{k-1} - q^{k-1-t}$
with $1 \leq t \leq \left\lfloor \frac{k-1}{2} \right\rfloor - 1$ for $k \geq 5$,
- (b) $q^{k-1} - q^{k-1-t} - q^t + 1 \leq d \leq q^{k-1}$ with $t = \left\lfloor \frac{k-1}{2} \right\rfloor$ for $k \geq 3$,

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

Much research on $n_q(k, d)$ has been done for small dimension k and small q by various methods. For $k = 3, 4, 5$ and $q = 3, 4, 5$, we can find tables of the values of $n_q(k, d)$ in [11] and [16].

To find the value of $n_q(k, d)$ for general q or k is more interesting. For minimum distance d with $q^{k-1} - q^{k-1-t} - q^t - sq + 1 \leq d \leq q^{k-1} -$

$q^{k-1-t} - q^t - (s-1)q$ for $1 \leq t \leq \lfloor \frac{k-1}{2} \rfloor$ and $1 \leq s \leq q-1$, which is just below the values for d in Corollary 1.3, it is known that there is no Griesmer code with

- (1) $t = 1$ and $s = 1$ for $q \geq 3, k \geq 5$ in [3, 4, 13, 14, 15],
- (2) $t = 1$ and $2 \leq s \leq q-1$ in [15],
- (3) $s = 1$ and $t = \lfloor \frac{k-1}{2} \rfloor$ for $k \geq 5$ in [3], $2 \leq t \leq \lfloor \frac{k-1}{2} \rfloor - 1$ for $k \geq 7$ in [5].

Naturally we can ask the cases $t \geq 2$ or $s \geq 2$. When $t = 2$ and $s = 2$, there is a result for the question for $k = 5$ in [2].

In this paper, we consider the case $t = 2$ and $s = 2$ for $k = 6$. In other words, we consider the problem whether Griesmer codes with minimum distance d with $q^5 - q^3 - q^2 - 2q + 1 \leq d \leq q^5 - q^3 - q^2 - q$ exist or not for $q \geq 4$. For $q = 2$ or 3 , we note that $n_2(6, d) = g_2(6, d)$ with $d = 17, 18$ ([6]) and $n_3(6, d) = g_3(6, d) + 1$ with $d = 202, 203, 204$ ([8, 17]).

As the first step to determine the exact value of $n_q(6, d)$ with $d = q^5 - q^3 - q^2 - 2q + \alpha, 1 \leq \alpha \leq q$ and $q \geq 4$, we need to prove the following.

THEOREM A. *There does not exist a $[g_q(6, d), 6, d]_q$ code with $d = q^5 - q^3 - q^2 - 2q + 1$ for $q \geq 4$.*

In Section 3, we give a proof of Theorem A and in Section 2, we recall some results needed to prove Theorem A.

Recall that the existence of an $[n, k, d]_q$ code with $d \geq 2$ implies the existence of an $[n-1, k, d-1]_q$ code. Therefore, by Theorem A, we have the following.

THEOREM B. *For $q \geq 4$, we have $n_q(6, d) \geq g_q(6, d) + 1$ with $q^5 - q^3 - q^2 - 2q + 1 \leq d \leq q^5 - q^3 - q^2 - q$.*

If we let $k = 6$ in Theorem 16 in [3], then we have the following.

THEOREM 1.4 ([3]). *For $q \geq 3$, there exists a $[g_q(6, d) + 1, 6, d]_q$ code for $q^5 - q^3 - 2q^2 + 1 \leq d \leq q^5 - q^3 - q^2$.*

By Theorem B and Theorem 1.4, we conclude the next theorem.

THEOREM C. *For $q \geq 4$, we have $n_q(6, d) = g_q(6, d) + 1$ with $q^5 - q^3 - q^2 - 2q + 1 \leq d \leq q^5 - q^3 - q^2 - q$.*

Finally, combining the result of [4] with Theorem C, for $q \geq 4$, we have the following:

$$n_q(6, d) = g_q(6, d) + 1 \text{ for } q^5 - q^3 - q^2 - 2q + 1 \leq d \leq q^5 - q^3 - q^2.$$

2. Preliminaries

Let $PG(r, q)$ be the r -dimensional projective space over \mathbb{F}_q and let θ_r be the number of points in $PG(r, q)$. Then $\theta_r = q^r + q^{r-1} + \cdots + q + 1$ for a positive integer r . For convenience, we let $\theta_0 = 1$ and $\theta_r = 0$ if $r < 0$. We call a subspace of dimension j in $PG(r, q)$ a j -flat. In particular, we call a subspace of dimension 0 [resp. 1, 2, $r - 1$] a point [resp. a line, a plane, a hyperplane].

Let C be a projective $[n, k, d]_q$ linear code with a generator matrix of G . Then no two columns of G are linearly dependent. Each column of G can be considered as a point of $PG(k - 1, q)$. Let C_1 be the set of all columns of G and let $C_0 = C_1^c$, the complement of C_1 in $PG(k - 1, q)$. For a subset S in $PG(k - 1, q)$, we use the following notation;

$$c_0(S) = |S \cap C_0|, \quad c(S) = |S \cap C_1| \quad \text{and} \quad c_0 = |C_0|.$$

In particular, for a projective $[n, 6, d]_q$ linear code C , we have $n = |C_1|$, $c_0 = \theta_5 - n$ and $d = n - \max\{c(H) \mid H \text{ is a 4-flat in } PG(5, q)\}$.

Now we recall theorems which play an important role to prove Theorem A.

For a subset S in the r -dimensional affine space $AG(r, q)$ over \mathbb{F}_q , S is a t -fold blocking set with respect to hyperplanes if every hyperplane in $AG(r, q)$ meets S in at least t points.

THEOREM 2.1 ([1]). *A t -fold blocking set S with respect to hyperplanes in $AG(r, q)$ satisfies*

$$|S| \geq (r + t - 1)(q - 1) + 1.$$

A subset F of $PG(r, q)$ with $|F| = f$ is called an $\{f, t; r, q\}$ -minihyper if every hyperplane meets F in at least t points. Hamada ([7]) showed that for $k \geq 3$ and $1 \leq d < q^{k-1}$, there is a one to one correspondence between the set of all nonequivalent $[n, k, d]_q$ Griesmer codes and the set of all $\{\theta_{k-1} - n, \theta_{k-2} - n + d; k - 1, q\}$ -minihypers. Thus an $[n, 6, d]_q$ Griesmer code C with $d < q^5$ corresponds to a $\{\theta_5 - n, \theta_4 - n + d; 5, q\}$ -minihyper. The following is a characterization of some minihypers.

THEOREM 2.2 ([7]). *Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be positive integers with $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_m \leq t$ and $1 \leq m \leq t$.*

- (a) *In the case $m = 1$, S is a $\{\theta_{\lambda_1}, \theta_{\lambda_1-1}; t, q\}$ -minihyper if and only if S is a λ_1 -flat in $PG(t, q)$.*
- (b) *In the case $m \geq 2$ and $t \geq \lambda_m + \lambda_{m-1} + 1$, S is a $\{\sum_{i=1}^m \theta_{\lambda_i}, \sum_{i=1}^m \theta_{\lambda_i-1}; t, q\}$ -minihyper if and only if S consists of disjoint union of λ_i -flats in $PG(t, q)$.*

- (c) *In the case $m \geq 2$ and $t \leq \lambda_m + \lambda_{m-1}$, there is no $\{\sum_{i=1}^m \theta_{\lambda_i}, \sum_{i=1}^m \theta_{\lambda_i-1}; t, q\}$ -minihyper.*

Let $m_{r,q}(s)$ denote the minimum value of f for which an $\{f, \theta_{r-2} + s; r, q\}$ -minihyper exists for $r \geq 3$ and $1 \leq s \leq q - 1$. If we let $r = 4$ and $s = 1$ or $s = 2$ in Theorem 2.4 in [15], then we have the following.

THEOREM 2.3 ([15]). *For $q \geq 3$, we have*

- (a) $m_{4,q}(1) \geq \theta_3 + \theta_1 + q$,
- (b) $m_{4,q}(2) \geq \theta_3 + 2\theta_1 + q$.

In [14], Maruta proved the nonexistence of a $[g_q(5, d), 5, d]_q$ code with $q^4 - 2q^2 - q + 1 \leq d \leq q^4 - 2q^2$ for $q \geq 3$. Landjev and Maruta [resp. Cheon *et al.*] proved the nonexistence of a $[g_q(5, d), 5, d]_q$ code with $q^4 - 2q^2 - 2q + 1 \leq d \leq q^4 - 2q^2 - q$ for $q = 4$ [resp. for $q \geq 5$] in [12] [resp. [2]]. Since those two intervals of d -values are consecutive, we conclude that there does not exist a Griesmer code with $q^4 - 2q^2 - 2q + 1 \leq d \leq q^4 - 2q^2$ for $q \geq 4$. Here we express the above results with the notion of minihyper respectively.

THEOREM 2.4 ([14]). *For $q \geq 3$ and $0 \leq e \leq q - 1$, there does not exist a $\{2\theta_2 + e, 2\theta_1; 4, q\}$ -minihyper.*

THEOREM 2.5 ([2, 12]). *For $q \geq 4$ and $0 \leq e \leq q - 1$, there does not exist a $\{2\theta_2 + \theta_1 + e, 2\theta_1 + 1; 4, q\}$ -minihyper.*

For a Griesmer code, the following holds:

THEOREM 2.6 ([14]). *Let C be a $[g_q(k, d), k, d]_q$ code and let $\gamma_j := \sum_{i=0}^j \left\lceil \frac{d}{q^{k-1-i}} \right\rceil$ for $0 \leq j \leq k - 2$. Then there exist j -flats Δ_j with $c(\Delta_j) = \gamma_j$ such that $\Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_{k-2}$ and that Δ_j gives a $[\gamma_j, j + 1, \gamma_j - \gamma_{j-1}]$ Griesmer code for $1 \leq j \leq k - 2$.*

If we let $t = 1$ in Theorem 7 in [10], then we have the following.

THEOREM 2.7. *For an integer $r \geq 0$ and $q \geq r + 1$, let S be a subset in $PG(m, q)$ with $|S| \leq \theta_{m-1} + r\theta_{m-3}$. If $|S \cap l| \geq 1$ for any line l in $PG(m, q)$ then S contains a hyperplane.*

3. Main theorem

In this section, we prove Theorem A. On the contrary, we assume that for $q \geq 4$, there exists a $[g_q(6, d), 6, d]_q$ code C with $d = q^5 - q^3 - q^2 - 2q + 1$.

Since C is a Griesmer code, by Theorem 2.6, we have the following:

$$\begin{aligned} c_0 &= \theta_3 + \theta_2 + 2q, \\ c_0(H) &\geq \theta_2 + \theta_1 + 1 \text{ for any 4-flat } H \text{ in } PG(5, q), \\ c_0(\Delta) &\geq \theta_1 + 1 \text{ for any 3-flat } \Delta \text{ in } PG(5, q), \\ c_0(\delta) &\geq 1 \text{ for any 2-flat } \delta \text{ in } PG(5, q). \end{aligned}$$

Let H_0 be a 4-flat in $PG(5, q)$ with $c_0(H_0) = \theta_2 + \theta_1 + 1$. Then H_0 is a $\{\theta_2 + \theta_1 + 1, \theta_1 + 1; 4, q\}$ -minihyper. By Theorem 2.2 (b), we note that $H_0 \cap C_0$ is a disjoint union of a plane δ_0 , a line l_0 and a point P_0 . For any 3-flat Δ in H_0 , we have

$$(3.1) \quad c_0(\Delta) = \theta_2 + 2, \quad \theta_2 + 1, \quad 2\theta_1 + 1, \quad 2\theta_1, \quad \theta_1 + 2, \quad \text{or} \quad \theta_1 + 1.$$

For any 4-flat H in $PG(5, q)$, we have $c_0(H_0 \cap H) \leq \theta_2 + 2$ by (3.1). Since $c_0(H) \geq \theta_2 + \theta_1 + 1$ for any 4-flat H , we have

$$\begin{aligned} c_0 &= c_0(H) + \sum_{H_0 \cap H \subseteq H' \neq H} c_0(H') - qc_0(H_0 \cap H) \\ &\geq c_0(H) + q(\theta_2 + \theta_1 + 1) - q(\theta_2 + 2) \\ &= c_0(H) + q^2, \end{aligned}$$

which implies $c_0(H) \leq \theta_3 + 2\theta_1 + q - 1$.

Therefore, we conclude that

$$(3.2) \quad \theta_2 + \theta_1 + 1 \leq c_0(H) \leq \theta_3 + 2\theta_1 + q - 1 \text{ for any 4-flat } H \text{ in } PG(5, q).$$

Now we will derive a contradiction in two steps as follows: In Step I, we prove that there is no 4-flat H such that

$$(3.3) \quad 2\theta_2 + 1 \leq c_0(H) \leq \theta_3 + 2\theta_2 + q - 1.$$

Then, by (3.2), we conclude that $\theta_2 + \theta_1 + 1 \leq c_0(H) \leq 2\theta_2$ for any 4-flat H in $PG(5, q)$. In Step II, we will prove that it is impossible.

Step I. We divide the interval (3.3) into five small intervals, which we refer to as Case 1, ..., Case 5 and we prove the nonexistence of a 4-flat H with $c_0(H)$ belonging to each small interval.

When we prove them we use the following computation frequently: For a 4-flat H_1 in $PG(5, q)$, let Δ be a 3-flat in H_1 . Then we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta \subseteq H \neq H_1} c_0(H) - qc_0(\Delta) \\ &\geq c_0(H_1) + q(\theta_2 + \theta_1 + 1) - qc_0(\Delta), \end{aligned}$$

and hence

$$(3.4) \quad c_0(\Delta) \geq \frac{c_0(H_1) - c_0}{q} + \theta_2 + \theta_1 + 1.$$

Case 1. There is no 4-flat H with $\theta_3 + 2\theta_1 \leq c_0(H) \leq \theta_3 + 2\theta_1 + q - 1$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = \theta_3 + 2\theta_1 + q - 1 - f$, $0 \leq f \leq q - 1$. Let Δ be a 3-flat in H_1 . Using (3.4), we have $c_0(\Delta) \geq \theta_2 + 2 - \frac{f}{q}$. Since $0 \leq f \leq q - 1$, we have

$$c_0(\Delta) \geq \theta_2 + 2 \text{ for all 3-flat } \Delta \subset H_1.$$

Furthermore, if we let $\Delta = H_0 \cap H_1$, then $c_0(\Delta) = \theta_2 + 2$ since $c_0(H_0 \cap H_1) \leq \theta_2 + 2$ by (3.1). Thus $C_0 \cap H_1$ is a $\{\theta_3 + 2\theta_1 + q - 1 - f, \theta_2 + 2; 4, q\}$ -minihyper, which contradicts Theorem 2.3 (b). \square

Case 2. There is no 4-flat H with $\theta_3 + \theta_1 + 1 \leq c_0(H) \leq \theta_3 + \theta_1 + q$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = \theta_3 + \theta_1 + q - f$, $0 \leq f \leq q - 1$. Let Δ be a 3-flat in H_1 . Using (3.4), we have $c_0(\Delta) \geq \theta_2 + 1$ since $0 \leq f \leq q - 1$.

Suppose that $c_0(\Delta) \geq \theta_2 + 2$ for all 3-flat $\Delta \subseteq H_1$. By Theorem 2.3 (b), we have $c_0(H_1) \geq \theta_3 + 2\theta_1 + q$ which is a contradiction. Thus there exists a 3-flat Δ with $c_0(\Delta) = \theta_2 + 1$. Then $C_0 \cap H_1$ is a $\{\theta_3 + \theta_1 + q - f, \theta_2 + 1; 4, q\}$ -minihyper. By Theorem 2.3 (a), we obtain $f = 0$, that is, $C_0 \cap H_1$ is a $\{\theta_3 + \theta_1 + q, \theta_2 + 1; 4, q\}$ -minihyper. Here, we prove the following claim.

Claim. $C_0 \cap H_1$ contains a 3-flat.

Proof of Claim: To prove Claim, it suffices to prove that for any line l in H_1 , $|(C_0 \cap H_1) \cap l| \geq 1$ by Theorem 2.7. Suppose that there is a line l_1 in H_1 with $|(C_0 \cap H_1) \cap l_1| = 0$. Consider a 3-flat Δ in H_1 containing l_1 . Then there is a 3-flat Δ' containing l_1 with $|(C_0 \cap H_1) \cap \Delta'| = \theta_2 + 1$ since $c_0(H_1) = \theta_3 + \theta_1 + q$ and

$$c_0(H_1) = \frac{1}{\theta_1} \left(\sum_{l_1 \subset \Delta \subset H_1} |(C_0 \cap H_1) \cap \Delta| \right) < \frac{1}{\theta_1} \cdot \theta_2 \cdot (\theta_2 + 2).$$

On the other hand, we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta' \subseteq H, H \neq H_1} c_0(H) - qc_0(\Delta') \\ &\geq c_0(H_1) + q(\theta_2 + \theta_1 + 1) - q(\theta_2 + 1) = \theta_3 + \theta_2 + 2q. \end{aligned}$$

Thus we note $c_0(H) = \theta_2 + \theta_1 + 1$ for any 4-flat $H (\neq H_1)$ containing Δ' . Let H_2 be a 4-flat containing Δ' with $c_0(H_2) = \theta_2 + \theta_1 + 1$. Then by Theorem 2.2 (b), $C_0 \cap H_2$ consists of a plane, a line and a point. Since $|(C_0 \cap H_2) \cap H_1| = |(C_0 \cap H_1) \cap \Delta'| = \theta_2 + 1$, it holds that $(C_0 \cap H_1) \cap \Delta'$ consists of a plane δ' and a point P' . Then $|(C_0 \cap H_1) \cap l_1| \geq |\delta' \cap l_1| \geq 1$, which is a contradiction to the choice of l_1 . Thus Claim is proved.

By Claim, $C_0 \cap H_1$ contains a 3-flat, say Δ_1 . Let $S = (C_0 \cap H_1) - \Delta_1$. We note $|S| = \theta_1 + q$. On the other hand, since $c_0(\Delta) \geq \theta_2 + 1$ for any 3-flat Δ in H_1 , $|S \cap \Delta| \geq 1$ for any 3-flat $\Delta \neq \Delta_1$ in H_1 . Thus S can be considered as 1-fold blocking set with respect to hyperplanes in $AG(4, q)$. By Theorem 2.1, we have $|S| \geq (4 + 1 - 1)(q - 1) + 1 = 4q - 3$ which is a contradiction since $q \geq 4$. \square

Case 3. There is no 4-flat H with $2\theta_2 + \theta_1 + q \leq c_0(H) \leq \theta_3 + \theta_1$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = \theta_3 + \theta_1 - eq - f$, $0 \leq f \leq q - 1$, $0 \leq e \leq q^2 - q - 3$. By (3.4), we have $c_0(\Delta) \geq \theta_2 - e$. Suppose there is a 3-flat Δ_1 with $c_0(\Delta_1) = \theta_2 - e$. Since $0 \leq e \leq q^2 - q - 3$, we have $2\theta_1 + 2 \leq c_0(\Delta_1) \leq \theta_2$. By (3.1), we note that $c_0(H) \geq \theta_2 + \theta_1 + 2$ for any 4-flat H containing Δ_1 . Thus we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) - qc_0(\Delta_1) \\ &\geq c_0(H_1) + q(\theta_2 + \theta_1 + 2) - q(\theta_2 - e) = \theta_3 + \theta_2 + 2q + \theta_1 - (f + 1), \end{aligned}$$

which is a contradiction since $0 \leq f \leq q - 1$. Thus $c_0(\Delta) \geq \theta_2 - e + 1$ for any 3-flat Δ in H_1 .

On the other hand, H_1 corresponds to an $[n_1, 5, d_1]_q$ linear code with $n_1 = \theta_4 - c_0(H_1) = \theta_4 - \theta_3 - \theta_1 + eq + f$ and $d_1 \geq q^4 - q^3 - q + eq + f - e$. Applying the Griesmer bound, we have

$$\begin{aligned} g_q(5, d_1) &\geq q^4 - q^3 - q + eq + f - e \\ &\quad + q^3 - q^2 - 1 + e + \left\lceil \frac{f - e}{q} \right\rceil \\ &\quad + q^2 - q + \left\lceil \frac{eq - q + f - e}{q^2} \right\rceil \end{aligned}$$

$$\begin{aligned}
 & + q - 1 + \left\lceil \frac{eq - q + f - e}{q^3} \right\rceil \\
 & + 1 \\
 \geq & n_1 + \left\lceil \frac{f - e}{q} \right\rceil + \left\lceil \frac{eq - q + f - e}{q^2} \right\rceil + \left\lceil \frac{eq - q + f - e}{q^3} \right\rceil.
 \end{aligned}$$

Let

$$T = \left\lceil \frac{f - e}{q} \right\rceil + \left\lceil \frac{eq - q + f - e}{q^2} \right\rceil + \left\lceil \frac{eq - q + f - e}{q^3} \right\rceil.$$

Then $T \leq 0$ by the Griesmer bound. Now we prove the following claim.

Claim. In the set of pairs (e, f) with $0 \leq e \leq q^2 - q - 3$ and $0 \leq f \leq q - 1$, we have the following:

$$T \leq 0 \text{ if and only if } (e, f) = (0, 0), (1, 0), \text{ or } (1, 1).$$

Moreover, in this case $T = 0$.

Proof of Claim: We prove

$$\begin{cases} T = 0, & \text{if } (e, f) = (0, 0), (1, 0), \text{ or } (1, 1), \\ T > 0, & \text{otherwise.} \end{cases}$$

When $(e, f) = (0, 0), (1, 0)$ or $(1, 1)$, we note that $T = 0$. Hence we consider the other case.

For $f \geq 2$, since $0 \leq e \leq q^2 - q - 3 < q^2$, we have

$$\begin{aligned}
 T & \geq \left\lceil \frac{f - e}{q} + \frac{eq - q + f - e}{q^2} + \frac{eq - q + f - e}{q^3} \right\rceil \\
 & = \left\lceil \frac{(f - 1)\theta_2 + 1 - e}{q^3} \right\rceil > 0.
 \end{aligned}$$

Now, consider the case $f = 0$ or 1 .

Assume $f = 1$. We have $eq - q + f - e = (e - 1)(q - 1)$. For $e = 0$, we have $T = \left\lceil \frac{1}{q} \right\rceil + \left\lceil \frac{-(q-1)}{q^2} \right\rceil + \left\lceil \frac{-(q-1)}{q^3} \right\rceil = 1 > 0$. For $2 \leq e \leq q$, we have $T = \left\lceil \frac{1-e}{q} \right\rceil + \left\lceil \frac{(e-1)(q-1)}{q^2} \right\rceil + \left\lceil \frac{(e-1)(q-1)}{q^3} \right\rceil = 0 + 1 + 1 = 2 > 0$. For $tq + 1 \leq e \leq (t + 1)q$ with $1 \leq t \leq q - 2$, we have $\left\lceil \frac{1-e}{q} \right\rceil = -t$, $t \leq \left\lceil \frac{(e-1)(q-1)}{q^2} \right\rceil \leq t + 1$ and $\left\lceil \frac{(e-1)(q-1)}{q^3} \right\rceil = 1$, and hence $T > 0$.

Finally, assume $f = 0$. For $2 \leq e \leq q - 1$, we get $T = \left\lceil \frac{-e}{q} \right\rceil + \left\lceil \frac{e(q-1)-q}{q^2} \right\rceil + \left\lceil \frac{e(q-1)-q}{q^3} \right\rceil = 0 + 1 + 1 = 2 > 0$. For $tq \leq e \leq (t + 1)q - 1$

with $1 \leq t \leq q-2$, we have $\left\lceil \frac{-e}{q} \right\rceil = -t$, $t \leq \left\lceil \frac{e(q-1)-q}{q^2} \right\rceil \leq t+1$ and $\left\lceil \frac{e(q-1)-q}{q^3} \right\rceil = 1$, and hence $T > 0$. Thus the claim is proved.

By the claim, there remain only three possibilities for $c_0(H_1)$.

If $(e, f) = (0, 0)$, then $C_0 \cap H_1$ is a $\{\theta_3 + \theta_1, \theta_2 + 1; 4, q\}$ -minihyper, which is impossible by Theorem 2.2 (c).

Assume $(e, f) = (1, 0)$. Then $C_0 \cap H_1$ is a $\{\theta_3 + 1, \theta_2; 4, q\}$ -minihyper. By Theorem 2.2 (b), the minihyper $C_0 \cap H_1$ consists of a 3-flat Δ_1 and a point P_1 . From $c_0 = c_0(H_1) + \sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) - qc_0(\Delta_1)$, since $c_0(\Delta_1) = \theta_3$, we have

$$\sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) = q(\theta_3 + \theta_1 + 2).$$

Hence there is a 4-flat H containing Δ_1 with $c_0(H) \geq \theta_3 + \theta_1 + 2$, which contradicts Case 1 and Case 2.

If $(e, f) = (1, 1)$ then $C_0 \cap H_1$ is a $\{\theta_3, \theta_2; 4, q\}$ -minihyper, i.e., a 3-flat. Then $c_0(H_0 \cap H_1) = \theta_2$ or θ_3 , which is a contradiction by (3.1). \square

Case 4. There is no 4-flat H with $2\theta_2 + \theta_1 \leq c_0(H) \leq 2\theta_2 + 2q$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = 2\theta_2 + 2q - f$, $0 \leq f \leq q-1$. For any 3-flat Δ in H_1 , by (3.4) we have $c_0(\Delta) \geq 2\theta_1 + 1$.

Suppose that $c_0(\Delta) \geq 2\theta_1 + 2$ for all $\Delta \subseteq H_1$. Then H_1 is an $[n_1, 5, d_1]_q$ code with $n_1 = \theta_4 - 2\theta_2 - 2q + f$ and $d_1 \geq q^4 - 2q^2 - 2q + f + 2$, which contradicts the Griesmer bound. Thus there is a 3-flat Δ in H_1 with $c_0(\Delta) = 2\theta_1 + 1$. Then $C_0 \cap H_1$ is a $\{2\theta_2 + 2q - f, 2\theta_1 + 1; 4, q\}$ -minihyper, which contradicts Theorem 2.5. \square

Case 5. There is no 4-flat H with $2\theta_2 + 1 \leq c_0(H) \leq 2\theta_2 + q$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = 2\theta_2 + q - f$, $0 \leq f \leq q-1$. For any 3-flat Δ in H_1 , we have $c_0(\Delta) \geq 2\theta_1$ by (3.4). Suppose that $c_0(\Delta) \geq 2\theta_1 + 1$ for any Δ in H_1 . Then H_1 corresponds to an $[n_1, 5, d_1]_q$ linear code with $n_1 = \theta_4 - 2\theta_2 - q + f$ and $d_1 \geq q^4 - 2q^2 - q + f + 1$, which contradicts the Griesmer bound. Thus there exists a 3-flat Δ in H_1 with $c_0(\Delta) = 2\theta_1$, and hence $C_0 \cap H_1$ is a $\{2\theta_2 + q - f, 2\theta_1; 4, q\}$ -minihyper. For $1 \leq f \leq q-1$, by Theorem 2.4, such a minihyper $C_0 \cap H_1$ does not exist. Thus we have $f = 0$ and

$C_0 \cap H_1$ is a $\{2\theta_2 + q, 2\theta_1; 4, q\}$ -minihyper. Let Δ_1 be a 3-flat in H_1 with $c_0(\Delta_1) = 2\theta_1$. Then we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) - qc_0(\Delta_1) \\ &= 2\theta_2 + q + \sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) - q \cdot 2\theta_1, \end{aligned}$$

and hence

$$\sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) = q(\theta_2 + \theta_1 + 1).$$

Thus we conclude $c_0(H) = \theta_2 + \theta_1 + 1$ for all 4-flat H ($\neq H_1$) containing Δ_1 , say H_2, \dots, H_{q+1} . By Theorem 2.2 (b), for $2 \leq i \leq q+1$, $H_i \cap C_0$ consists of a plane δ_i , a line l_i and a point Q_i , respectively. Let $C_0 \cap \Delta_1 = C_0 \cap H_1 \cap H_i = l_i \cup m_i$, $2 \leq i \leq q+1$, where $m_i = \delta_i \cap H_1$ is a line. Since $c_0(\Delta_1) = 2\theta_1$ and $q \geq 4$, there exist H_i and H_j such that $m_i = m_j$, $2 \leq i < j \leq q+1$. Then we note that $\delta_i \cap \delta_j = m_i$. Consider the linear span of δ_i and δ_j , denoted by $\langle \delta_i, \delta_j \rangle$. Then $c_0(\langle \delta_i, \delta_j \rangle) \geq \theta_2 + q^2$. Thus we have

$$\begin{aligned} c_0 &= \sum_{\langle \delta_i, \delta_j \rangle \subseteq H} c_0(H) - qc_0(\langle \delta_i, \delta_j \rangle) \\ &\leq \sum_{\langle \delta_i, \delta_j \rangle \subseteq H} c_0(H) - q(\theta_2 + q^2), \end{aligned}$$

and hence $\sum_{\langle \delta_i, \delta_j \rangle \subseteq H} c_0(H) \geq 3\theta_3 + 2q - 1$. Therefore, there exists a 4-flat H containing $\langle \delta_i, \delta_j \rangle$ with $c_0(H) \geq 3q^2 + 5$. However, from Case 1, 2, 3, 4, there is no 4-flat H with $2\theta_2 + \theta_1 \leq c_0(H) \leq \theta_3 + 2\theta_1 + q - 1$. Thus we have a contradiction since $q \geq 4$. \square

Step II. By Step I and (2) we conclude that

$$(3.5) \quad \theta_2 + \theta_1 + 1 \leq c_0(H) \leq 2\theta_2 \quad \text{for any 4-flat } H \text{ in } PG(5, q).$$

On the other hand, since $H_0 \cap C_0$ is a disjoint union of a plane δ_0 , a line l_0 and a point P_0 , the linear span $\Delta_0 = \langle \delta_0, P_0 \rangle$ is 3-flat with $c_0(\Delta_0) = \theta_2 + 2$. Then we have

$$\begin{aligned}
c_0 &= c_0(H_0) + \sum_{\Delta_0 \subseteq H, H \neq H_0} c_0(H) - qc_0(\Delta_0) \\
&= \theta_2 + \theta_1 + 1 + \sum_{\Delta_0 \subseteq H, H \neq H_0} c_0(H) - q(\theta_2 + 2),
\end{aligned}$$

and hence $\sum_{\Delta_0 \subseteq H, H \neq H_0} c_0(H) = 2\theta_3 + 3q - 3$. Thus there is a 4-flat $H (\neq H_0)$ containing Δ_0 with $c_0(H) \geq 2\theta_2 + 3$, which contradicts (3.5). Thus the proof is completed. \square

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